Geometry of almost-product Lorentzian manifolds and relativistic observer

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Abstract

The notion of relativistic observer is confronted with Naveira's classification of (pseudo-)Riemannian almost-product structures on space-time manifolds. Some physical properties and their geometrical counterparts are shortly discussed.

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1 Introduction

In Einstein's General Relativity a gravitational interaction is represented by a metric with Lorentzian signature $(-,+,+,+)$ living on a (curved) four-dimensional space-time manifold and satisfying Einstein's field equations. An observer is independent notion and according to a nowadays point of view it can be identified with an arrow of time. More precisely, the observer is determined by a time-like normalized (local) vector field on space-time. We can also think about it as a collection of its integral curves considered as world lines (also known as a congruence of world lines of point observers) of some continuous material object (e.g. relativistic fluid). From mathematical perspective it provides a one-dimensional (time-like) foliation. It appears that a pair: the metric and the vector field determines a differential-geometric structure which is called an almost-product structure. From physical perspective a relativistic observer is tautologically defined as field of his own four-velocities. Having chosen observer one can define relativistic observables, i.e. relative measurable quantities. They include relative (three-)velocity of another observer or test particles (see e.g. [1],[2]) as well as Noether conserved currents in diffeomorphism covariant field theories [3]. The well-know splitting of electromagnetic field into measurable electric and magnetic components is also relative to the observer. In the more traditional approach to General Relativity the measurable quantities are related to coordinates. In fact, given coordinate system one can associate to it a (local) observer indicated by a time variable. However the notion adopted here is more general, coordinate-free and can be globalized.

In the presented note we provide the correspondence between Naveira's classes of a pseudo-Riemannian manifold [4] implemented by the observer and its physical characteristics as introduced in [5].

The paper is organized as follows. In section 2 we introduce the notation and basic notions. In section 3 we shortly recall Gil-Medrano's theorem [6] which provides a differential geometric interpretation for Naveira's classes. The advantages of the almost-product structures in physics are discussed in section 4 (see also [7] in this context). They extend possible characteristics for a given observer on Lorentzian manifold. Finally, we provide few illustrative examples in section 5.

2 Preliminaries and definitions

Let M and TM denote respectively n-dimensional smooth manifold and its tangent bundle. A k-dimensional ($k < n$) tangent distribution (k-distribution in short) is a map D which associates a k-dimensional subspace $D_p \subset T_pM$ with the point $p \in M$:

$$D:\ p \ \rightarrow\ D_p \subset T_pM. \tag{1}$$

D can be also considered as a subbundle of TM. Locally, one can say that k-distribution is generated by a set of k linearly independent vector fields $X_i \in \Gamma(TM),\ i = 1,2,...,k$ iff in every point p their

values span the k-dimensional subspace D_p, i.e. $D_p = \mathrm{span}\{X_1(p), \ldots, X_k(p)\}$. In this case we shall write $X_i \in \Gamma(D)$.

An embedded submanifold $N \subset M$ is called an integral manifold of the distribution D if $T_pN = D_p$ in every point $p \in N$. We say that D is involutive if for each pair of local vector fields (X, Y) belonging to D their Lie bracket $[X, Y]$ is also a vector field from D.

The distribution D is completely integrable if for each point $p \in M$ there exists an integral manifold N of the distribution D passing through p such the dimension of N is equal to the dimension of D. It turns out that every involutive distribution is completely integrable (local Frobenius theorem). Every smooth 1-dimensional distribution is integrable.

An integrability of a distribution is closely related to the notion of a foliation. We have the following (global) Frobenius theorem:

Theorem 1. *Let D be an involutive k-dimensional tangent distribution on a smooth manifold M. The collection of all maximal connected integral manifolds of D forms a foliation of M.*

The proof of the theorem and the precise definition of a foliation can be found in [8]. Roughly speaking, a foliation is a collection of submanifolds N_i so that one submanifold proceeds smoothly into another one. They do not cross each other. Particularly, a class of globally hyperbolic space-times $M = T \times \Sigma$, where T is an open interval in the real line $\mathbb{R}$ and Σ is a three-manifold, serve as a typical example of global foliation [9].

Let us recall [10, 11] that an almost-product structure on M is determined by a field of endomorphisms of TM, i.e. $(1, 1)$ tensor field P on M, such that $P^2 = I$ (I =identity). In this case at any point $p \in M$ one can consider two subspaces of T_pM corresponding respectively to two eigenvalues ± 1 of P. It defines two complementary distributions on M, i.e. $TM = D^+ \oplus D^-$. Moreover, if M is equipped with a (pseudo)-Riemanian metric g such that:

$$g(PX, PY) = g(X, Y), \quad X, Y \in \Gamma(TM), \tag{2}$$

then both distributions are mutually orthogonal. In this case P is called a (pseudo -) Riemaniann almost-product structure. It is to be noticed that some modified gravity models admit almost-product structures as a solution [12].

3 Geometric characterization of distributions on (pseudo-)Riemannian manifold

Let D be a distribution on (M, g) and $D^\perp$ the distribution orthogonal to D. At every point $p \in M$, we have then $T_mM = D_m \oplus D_m^\perp$ [1]. Thus we can uniquely define a $(1, 1)$ tensor field P such that

[1] The case of null distributions is more complicated and should be discussed separately, see e.g. [13].

$P^2 = I,\ P_{|D} = 1,\ P_{|D^\perp} = -1$. It is clear that P becomes automatically a (pseudo -) Riemaniann almost-product structure. One has (see [6]):

Definition 1. *The distribution D is called: geodesic, minimal or umbilical if and only if D has property D_1, D_2, D_3 respectively, where:*

- $D_1 \iff (\nabla_A P)A = 0,$
- $D_2 \iff \alpha(X) = 0,$
- $D_3 \iff g((\nabla_A P)B, X) + g((\nabla_B P)A, X) = \frac{2}{k} g(A, B)\alpha(X),$

where $X \in \Gamma(D)^\perp$, $A, B \in \Gamma(D)$. Here $\{e_a\}_{a=1}^k$, $(k = \mathrm{dim}D)$ is a local orthonormal frame of D and $\alpha(X) = \sum_{a=1}^{k} g((\nabla_{e_a} P)e_a, X)$.

It implies that a distribution has the property D_1 if and only if it has the properties D_2 and D_3. Their meanings in the case of integrability are explained below.

Theorem 2. *(O.Gil-Medrano) A foliation D is a totally geodesic, minimal or totally umbilical if and only if D has the property F_1, F_2, F_3 respectively, where*

$$F_i \iff F + D_i, \quad i = 1, 2, 3 \tag{3}$$

and

$$F \iff (\nabla_A P)B = (\nabla_B P)A \ \ \forall A, B \in \Gamma(D). \tag{4}$$

The proof of this theorem can be found in [6]. It is easy to see that the property F is equivalent to Frobenius' theorem, i.e. the distribution D with this property is a maximal foliation. The theorem says that, in principle, one deals with three special types of foliations:

(F_1) Totally geodesic foliation: it means that every geodesic of an arbitrary integral submanifold N (the leaf of foliation) if considered together with induced metric (the first fundamental form) is at the same time geodesic of the total manifold M. Moreover, it is equivalent with the statement that the second fundamental form of N (i.e. extrinsic curvature) vanishes. In other words the extrinsic curvature measures the failure of a geodesic of the manifold N to be a geodesic of M.

(F_2) Minimal foliation. If there is a surface with the smallest possible value of area bounded by a certain curve that surface is called a minimal surface. The condition for a distribution to be a minimal distribution is that a trace of the second fundamental form vanishes. The trace of the extrinsic curvature is also called a mean curvature that is the average of the principal curvatures. Examples of minimal surfaces in $\mathbb{R}^3$ are a catenoid and helicoid.

(F_3) Umbilical foliation. We recall that umbilical manifold is a manifold for which all points are umbilical points. Umbilical points in turn are locally spherical: every tangent vector at such point is a principal direction and all principal curvatures are equal [14]. For example a sphere is umbilical manifold. In the case of integral submanifolds the second fundamental form has to be proportional to the induced metric.

4 Almost-product structure related to space-time observer

In the present section we are going to apply the formalism presented above to the special case of relativistic observer on a space-time manifold. These new tools will be used at the end of the section for a final classification.

From now on (M, g) denotes a four-dimensional manifold (space-time) equipped with Lorentzian signature metric $g_{\alpha\beta}$. An observer is represented by time-like vector field u^α which according to our sign convention $(-,+,+,+)$ is normalized to

$$u^\alpha u_\alpha = -1. \tag{5}$$

Strictly speaking the normalization condition (5) prevents existence of critical points and one can deal with one-dimensional (time-like) distribution instead. Such distribution is always integrable and provides a foliation with world-lines as leaves. Each leaf can be then parameterized by arc length (proper time) making u^α a four-velocity field. This implies that the only nontrivial question one can ask about one-dimensional distribution is weather or not it is geodesic (see below).

Because of this one should concentrate on its orthogonal (transversal) completion D. This is space-like three-dimensional distribution with Euclidean signature. Both distributions provide a 3+1 (orthogonal) decomposition of the tangent bundle $TM = D \oplus D^\perp$ with the one-dimensional time-like distribution denoted as $D^\perp$. It is easy to find out that the corresponding three-dimensional projection tensor has the form:

$$h^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta, \tag{6}$$

which, due to (5), implies $h^\alpha_\rho h^\rho_\beta = h^\alpha_\beta$. We would like to stress that in what follows we shall always use the original metric $g_{\alpha\beta}$ for lowering and rising indices. Thus covariant and contravariant components of tensors can be used exchangeable. For example the second-rank symmetric tensor

$$h_{\alpha\,\beta} = g_{\alpha\,\beta} + u_\alpha u_\beta, \tag{7}$$

plays the role of induced Euclidean metric on the distribution D. When D is integrable then (7) is the first fundamental form (i.e. induced metric) on each leaf. The corresponding foliation by space-like hypersurfaces has the physical meaning of clock synchronization and divides space-time into equal-time pieces identified with three-dimensional spaces. One should mention that the integrability of D is

always required in a case of $3+1$ splitting which is necessary for the Hamiltonian formalism of General Relativity (see e.g. [15]).

More generally to any tensor $A^{\alpha\cdots}_{\beta\cdots}$ living in space-time one can assign its projected three-dimensional counterpart

$$\tilde{A}^{\alpha\cdots}_{\beta\cdots} = h^{\alpha}_{\mu} h^{\nu}_{\beta} \cdots A^{\mu\cdots}_{\nu\cdots} \tag{8}$$

According to commonly accepted ideas, only projected three-dimensional tensors are good candidates for measurable relativistic observables. Obviously, such quantities are relative, i.e. observer dependent. For example for anti-symmetric covariant two-tensor $F_{\alpha\beta} = -F_{\beta\alpha}$ (two form), which under the closeness condition ($dF = 0$) can be interpreted as an electromagnetic field, one gets

$$F_{\alpha\beta} = H_{\alpha\beta} + u_{\alpha} E_{\beta} - u_{\beta} E_{\alpha} \tag{9}$$

where $H_{\alpha\beta} = \tilde{F}_{\alpha\beta} = h^{\mu}_{\alpha} h^{\nu}_{\beta} F_{\mu\nu}$ and $E_{\alpha} = u^{\mu} h^{\nu}_{\alpha} F_{\mu\nu}$ are measurable electric and magnetic components.

Before proceeding further let us answer the question when the one-dimensional foliation spanned by u is totally geodesic. This can be easily done by studding auto-parallel (geodesic) equation

$$u^{\beta} u_{\alpha;\beta} = 0 \tag{10}$$

where $u_{\alpha;\beta} = \nabla_{\beta} u_{\alpha}$ denotes the Levi-Civita covariant derivative of u. Thus introducing the acceleration vector $\dot{u}_{\alpha} = u^{\beta} u_{\alpha;\beta}$ one can conclude that vanishing of $\dot{u}^{\alpha}$ is equivalent to the geodesic equation (10). One should notice that $\dot{u}_{\alpha}$ is, in fact, a three-vector since $u^{\alpha}\dot{u}_{\alpha} = 0$.

In general, one can decompose the space components of the two-tensor $u_{\alpha;\sigma}$ into irreducible with respect to three-dimensional orthogonal group parts:

$$\tilde{u}_{\alpha;\beta} = h^{\sigma}_{\beta} u_{\alpha;\sigma} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta}, \tag{11}$$

where $\omega_{\alpha\beta}$ denotes its antisymmetric part, $\sigma_{\alpha\beta}$ is traceless symmetric component and finally Θ stands for the trace. This is a kinematical decomposition [2]. Using (6) we shall obtain:

$$u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} - \dot{u}_{\alpha} u_{\beta}. \tag{12}$$

There is a well-known interpretation of observer in terms of relativistic hydrodynamics treating him as a flow of material points constituting (perfect) fluid (continuous medium) with the world lines being particles trajectories: one line of the flow passes through every point x^{α} of a certain space-like (possibly bounded) region in a space-time. Accordingly, the tensor $u_{\alpha;\sigma}$ determines the rate of change in the position of one material point with respect to the other one [18].

[2] The dynamical equation is known as Raychandhuri equation (see e.g. [16, 17])

Keeping in mind this fluid analogy, each irreducible component of the projected tensor $u_{\alpha;\beta}$ admits a physical interpretation which is contained in a self-explaining and intuitive names (for more detailed explanations see e.g. [5, 18, 19]). In more explicit form one has to take into account the following three-dimensional quantities:

$$\omega_{\alpha\beta} = u_{[\alpha;\beta]} + \dot{u}_{[\alpha}u_{\beta]} \qquad \text{is a rotation tensor,} \tag{13}$$

$$\sigma_{\alpha\beta} = u_{(\alpha;\beta)} + \dot{u}_{(\alpha}u_{\beta)} - \frac{1}{3}\Theta h_{\alpha\beta} \qquad \text{is a shear tensor,} \tag{14}$$

$$\Theta = u^{\alpha}_{;\alpha} \qquad \text{is a scalar of expansion,} \tag{15}$$

$$\dot{u}^{\alpha} = u^{\beta}u^{\alpha}_{;\beta} \qquad \text{is an acceleration vector.} \tag{16}$$

It is more convenient to use the scalars:

$$\dot{u} \equiv (\dot{u}_{\alpha}\dot{u}^{\alpha})^{\frac{1}{2}}, \quad \omega \equiv (\frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta})^{\frac{1}{2}}, \quad \sigma \equiv (\frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta})^{\frac{1}{2}}. \tag{17}$$

These are non-negative and vanish at the same time as their corresponding tensors. An observer is rotation-free, shear-free or expansion-free when $\omega = 0$, $\sigma = 0$, $\Theta = 0$ respectively. If all quantities vanish, then the observer is called rigid.

It is worth mentioning that the observer (four-velocity) field u^{α} can be used to construct energy momentum tensor of an ideal (incompressible) fluid

$$T_{\alpha\beta} = (p+\rho)\,u_{\alpha}u_{\beta} + pg_{\alpha\beta} \tag{18}$$

where a matter density ρ and a pressure p are fluid internal parameters determining its thermodynamical behavior. The same energy-moment tensor treated on the right-hand side of Einstein's equations as a source of the gravitational field influences on the metric. It suggests possible relationships between metric and fluid observer which are interesting subject for future research (see e.g. [17, 20]).

Now we are ready to classify all almost-product structures related to relativistic observers in gravitational space-times. As we have already mentioned the tensor h^{α}_{β} projects on a three-dimensional subspace while $-u^{\alpha}u_{\beta} = \delta^{\alpha}_{\beta} - h^{\alpha}_{\beta}$ projects on one-dimensional complementary distribution spanned by u^{α}. It turns out that the difference:

$$P^{\alpha}_{\beta} = h^{\alpha}_{\beta} - (-u^{\alpha}u^{\beta}) = \delta^{\alpha}_{\beta} + 2u^{\alpha}u_{\beta}. \tag{19}$$

represents an almost-product structure compatible with the metric g[3]. Now the almost-product structure (19) can be used to encode the observer u^{α}.

Since the issue of one-dimensional distribution have been already solved we should concentrate on the

[3]It easy to see that P satisfies the conditions $P^2 = I$ as well as (2).

three-dimensional one. One has 4 conditions to be imposed on P (see Definition 1 and Theorem 2). The umbilical case D_3, after some manipulations, produces:

$$u_{(\alpha;\beta)} + \dot{u}_{(\alpha} u_{\beta)} = \frac{1}{3} \sum_{i=1}^{3} e_i^{\beta} u_{\alpha;\beta} e_i^{\alpha}. \tag{20}$$

Here $\{e_i\}_{i=1}^3$ denotes local orthonormal frame of D. We notice that the sum of the right hand side is a three-dimensional trace of the tensor $u_{\alpha;\beta}$ thus D_3 is equivalent to vanishing of shear tensor.

The condition D_2 (a minimal distribution) leads to:

$$\sum_{i=1}^{3} e_i^{\beta} u_{\alpha;\beta} e_i^{\alpha} = 0 \tag{21}$$

which is equivalent to vanishing a scalar of expansion.

For a geodesic distribution (D_1) one obtains:

$$u_{(\alpha;\beta)} + \dot{u}_{(\alpha} u_{\beta)} = 0 \tag{22}$$

which is equivalent to vanishing of both characteristics: shear and expansion.

Similarly, one can show that the integrability condition (F) reduces to:

$$u_{[\alpha;\beta]} + \dot{u}_{[\alpha} u_{\beta]} = 0 \tag{23}$$

which means vanishing of a rotation.

The final results are contained in the tables. The first one concerns accelerated $\dot{u} \neq 0$ observers with all possibilities for the three-dimensional distribution taken into account. Similarly, in the table 2 one concerns free-falling observers ($\dot{u} = 0$). The bracket $(1, 3)$ in the first column indicates that the first symbol is for one-dimensional distribution and the second for the three-dimensional one. The tables also contain for each class of almost-product Lorentzian manifold its the physical interpretations.

5 Illustrative examples

5.1 Minkowski space-time

The most extreme case in Naveira's classification is (F_1, F_1) class, i.e. both distributions are totally geodesic foliations. In Minkowski space-time the metric is flat in Cartesian coordinate system ($\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$) so one can replace the covariant derivatives with partial ones. Then (F_1, F_1) case

Class (1,3)	Accelerated observers $\dot{u} \neq 0$	**Physical meaning**	**3-distribution**
$(F,-)$	$u_{\alpha;\beta} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} - \dot{u}_\alpha u_\beta$	——	non-integrable **distribution**
(F, D_2)	$\Theta = 0 \;\Rightarrow\; u_{\alpha;\beta} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} - \dot{u}_\alpha u_\beta$	expansion-free	minimal
(F, D_3)	$\sigma = 0 \;\Rightarrow u_{\alpha;\beta} = \omega_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} - \dot{u}_\alpha u_\beta$	shear-free	umbilical
(F, D_1)	$\Theta = \sigma = 0 \;\Rightarrow\; u_{\alpha;\beta} = \omega_{\alpha\beta} - \dot{u}_\alpha u_\beta$	shear-free & expansion-free	geodesic
(F, F)	$\omega = 0 \;\Rightarrow\; u_{\alpha;\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} - \dot{u}_\alpha u_\beta$	rotation-free	**foliation**
(F, F_2)	$\omega = \Theta = 0 \;\Rightarrow\; u_{\alpha;\beta} = \sigma_{\alpha\beta} - \dot{u}_\alpha u_\beta$	rotation-free & expansion-free	minimal
(F, F_3)	$\omega = \sigma = 0 \;\Rightarrow\; u_{\alpha;\beta} = \frac{1}{3}\Theta h_{\alpha\beta} - \dot{u}_\alpha u_\beta$	rotation-free & shear-free	totally umbilical
(F, F_1)	$u_{\alpha;\beta} = -\dot{u}_\alpha u_\beta$	rigid	totally geodesic

Class (1,3)	Geodesic (free falling) observers $\dot{u} = 0$	**Physical meaning**	**3-distribution**
$(F_1,-)$	$u_{\alpha;\beta} = u_{[\alpha;\beta]} + \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta}$	geodesic	non-integrable **distribution**
(F_1, D_2)	$\Theta = 0 \;\Rightarrow\; u_{\alpha;\beta} = u_{[\alpha;\beta]} + \sigma_{\alpha\beta}$	geodesic expansion-free	minimal
(F_1, D_3)	$\sigma = 0 \;\Rightarrow\; u_{\alpha;\beta} = u_{[\alpha;\beta]} + \frac{1}{3}\Theta h_{\alpha\beta}$	geodesic shear-free	umbilical
(F_1, D_1)	$\sigma = \Theta = 0 \;\Rightarrow\; u_{\alpha;\beta} = u_{[\alpha;\beta]}$	geodesic shear-free & expansion-free	geodesic
(F_1, F)	$\omega = 0 \;\Rightarrow\; u_{\alpha;\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta}$	geodesic rotation-free	**foliation**
(F_1, F_2)	$\omega = \Theta = 0, \;\Rightarrow\; u_{\alpha;\beta} = \sigma_{\alpha\beta}$	geodesic rotation-free & expansion-free	minimal
(F_1, F_3)	$\omega = \sigma = 0 \;\Rightarrow\; u_{\alpha;\beta} = \frac{1}{3}\Theta h_{\alpha\beta}$	geodesic rotation-free & shear-free	totally umbilical
(F_1, F_1)	$u_{\alpha;\beta} = 0$	geodesic rigid	totally geodesic

becomes just $u_{\alpha,\beta} = 0$. There exist a solution in a form of a constant vector. In fact, any constant time-like vector field can be changed by a linear transformation of coordinates (i.e. Lorentz transformation) into:

$$u^\alpha = [1, 0, 0, 0]\,. \tag{24}$$

It is a canonical inertial observer in Minkowski space-time. Less restrictive class is (F, F_1) what implies that the observer should accelerate. The example of such a observer is Rindler's one:

$$u^\alpha = \left[\frac{x}{x^2 - t^2}, \frac{t}{x^2 - t^2}, 0, 0\right], \tag{25}$$

for whom only a part of Minkowski space is available. The only non-vanishing characteristic is acceleration:

$$\dot{u} = (x^2 - t^2)^{-1/2}, \tag{26}$$

which is a constant along each trajectory. Again by introduction adapted (Rindler's) coordinates one can simplify expressions.

Let us consider the rotating observer in (x, y) plane:

$$u^\alpha = \left[\sqrt{2}, \frac{-y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}}, 0\right] \tag{27}$$

belonging to the class (F, D_2) with the following characteristics:

$$\Theta = 0, \tag{28}$$

$$\dot{u} = (x^2 + y^2)^{-1/2}, \tag{29}$$

$$\omega = \frac{\sqrt{2}}{2}(x^2 + y^2)^{-1/2}, \tag{30}$$

$$\sigma = \frac{\sqrt{2}}{2}(x^2 + y^2)^{-1/2} \tag{31}$$

constant along particles trajectories. The last example for Minkowski space-time (in the spherical coordinates $g_{\mu\nu} = \mathrm{diag}(-1, 1, r^2, r^2 \sin^2\theta)$) is the observer:

$$u^\alpha = \left[\frac{r}{\sqrt{r^2 - t^2}}, \frac{t}{\sqrt{r^2 - t^2}}, 0, 0\right]. \tag{32}$$

It turns out that the observer (32) has the following characteristics:

$$\dot{u} = (r^2 - t^2)^{-1/2}, \tag{33}$$

$$\omega = 0, \tag{34}$$

$$\Theta = \frac{2t}{r}(r^2 - t^2)^{-1/2}, \tag{35}$$

$$\sigma = \frac{\sqrt{3}t}{3r}(r^2 - t^2)^{-1/2} \tag{36}$$

and belongs to the class (F, F).

5.2 Schwarzschild space-time

There is no observer belonging to the class (F_1, F_1) in Schwarzschild space-time. It should satisfy the sixteen equations $u_{\alpha;\beta} = 0$ which turn out to be inconsistent.

Let us consider the observer:

$$u^\alpha = \left[(1 - \frac{2M}{r})^{-\frac{1}{2}}, 0, 0, 0\right]. \tag{37}$$

The only non-vanishing characteristic is the acceleration $\dot{u} = \frac{M}{r^2}(1 - \frac{2M}{r})^{-1/2}$ what implies that the observer (37) belongs to the class (F, F_1).
Geodesic observer [16] of the form:

$$u^\alpha = \left[(1 - \frac{3M}{r})^{-1/2}, 0, \sqrt{\frac{M}{r^2(r - 3M)}}, 0\right] \tag{38}$$

has the singular expansion at the north and south poles:

$$\Theta = \sqrt{\frac{M}{r^2(r - 3M)}}\cot\theta. \tag{39}$$

The scalars of the rotation and shear are:

$$\omega = \frac{1}{4}\sqrt{\frac{M}{r^3}}\left(\frac{1 - 6M/r}{1 - 3M/r}\right), \tag{40}$$

$$\sigma = \sqrt{\frac{27M\sin^2\theta(r - 2M)^2 + 16Mr(r - 3M)\cos^2\theta}{48r^3(r - 3M)^2\sin^2\theta}}. \tag{41}$$

He belongs to the $(F, -)$ class so due to this observer as well as (27) of Minkowski space-time there is no the three-dimensional orthogonal distribution as a providing foliation of the space-time manifold (M, g) i.e. there is no three-dimensional equal-time subspaces relative to those observers.

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